Credit Risk

Financial institutions are increasingly measuring and managing the risk from credit exposures at the portfolio level. The traditional binary classification in “good” and “bad” credits is not sufficient anymore, any credit can potentially become “bad” over time given the influences go towards an undesired direction.

But what is the credit risk of a given portfolio? Here comes one of many possible definitions:

Definition (Credit Risk or Credit Loss)
Credit Risk is the **distribution of financial losses** due to unexpected changes in the credit quality of a counterparty.

The distribution of credit losses is complex. At its center stands the **probability of default** (= likelihood of any type of failure to honor financial agreements) but we also have to address **credit exposures** (= how large will the outstanding obligation be, if default occurs?) and **recovery rates** (= what fraction of the exposure may be recovered?). We define each as random processes. The Credit Loss distribution is then defined as a product of these three distributions.

Let us describe the Credit Loss distribution more formally. Therefore we have to introduce some notation:

$$A := \{0,1,\ldots,T\},\quad B := \{j \in Nat : 1 \le \mathrm{j} \le \mathrm{J},\ \mathrm{J} \in \mathrm{Nat}\}, \tau \text{ is the default time of an obligor}$$

$$1_A^j(t): \textit{The "default" indicator function}$$

$$1_A^j(t) = \begin{cases} 1, \text{if } t \in A \text{ and obligor } j \text{ has already defaulted} \\ 0, \text{if } t \in A \text{ and obligor } j \text{ hasn't defaulted yet} \\ \text{undefined, if } t \notin A \end{cases}$$

$$\mathrm{E}_j(t): \textit{The "exposure at default" (EAD) function}$$
$$Range(E_j(t)) = \text{Total amount of payment obligations of obligor j given default at time t}$$

$$\mathrm{R}_j(t): \textit{The "recovery rate" function}$$
$$Range(R_j(t)) = \text{Proportion of } \mathrm{E}_j(t) \text{ that may be recovered in bankruptcy proceedings}$$

The portfolio loss D(T) being T the last time unit in our fixed time horizon is the sum of the individual losses $D_j(T)$ [B is the set of obligors]:

$$D(T)=\sum_{j\in B} D_j(T)=\sum_{j\in B} 1_A(T)\bullet E_j(\tau)\bullet R_j(\tau)$$

The individual credit losses $D_j(T)$ are unknown in advance. Because the sum of random variables is a random variable too is the full portfolio credit loss a random variable. The portfolio's credit loss distribution is the CDF of this random variable: (CDF=Cumulative Probability Distribution)

F(x) = Prob [D(T) <= x].

What we haven't modeled here were the correlations between the different $D_j(T)$. The correlation parameters typically have a strong influence on the tail of the Credit Loss distribution – and thus on the Value-at-Risk.

Portfolio Credit Loss Distributions have some interesting features:

i. The distribution is not symmetrical,
ii. The distribution is highly skewed, and
iii. The distribution has heavy tails.

We set up a stochastic model for calculating the Credit Loss Distribution. But how do we estimate the Credit Risk in practice?

Often financial institutions calculate the Credit Value-at-Risk (CVaR). This calculation assigns a single number to a portfolio, and this number is seen as the Economic Credit Capital.

By making the assumptions:

- The Probability of Default (within the time horizon T) Q(T) is for each credit the same, and
- The pair wise correlation rho of any two credits are always identical

we can calculate the CVaR for a confidence level X. That means we can be X% sure that the Credit Loss in T periods will not exceed the CVaR. A formula for calculating CVaR is given by

$$CVaR = L*(1-R)*V(X,T)$$

where

$$V(X,T)=\Phi\left(\frac{\sqrt{\rho}*\Phi^{-1}(X)+\Phi^{-1}(Q(T))}{\sqrt{1-\rho}}\right)$$

and

L = size of the portfolio,
R = recovery rate, and
rho = estimated correlation.

This formula was first established by Vasicek in “Probability of Loss on a Loan Portfolio”.

For internal purposes quite a lot of banks have established their own methodology of calculating CVaR. Here we will discuss CreditRisk+, a purely actuarial model where the risk drivers are the expected default rates.

In this model default is modeled as a stopping time random variable and follows a Poisson distribution.

We make the following assumptions:

i. The likelihood of default of a loan is in each period of time the same,
ii. The number of defaults that occur in any given period is independent of the number of defaults that occur in any other period

Under these assumptions, the PDF for the number of defaults in a given period of time (say one year) is given by a Poisson distribution:

$$Pr(n) = \frac{\lambda^n * exp(-\lambda)}{n!}$$

where

n = number of defaults in the holding period, and
λ = average number of defaults in the holding period

As we expect the average default rate to change over time (business cycle!), the Poisson distribution can only be used to represent the default process.

In order to derive the Loss distribution the exposures (net of the corresponding recovery rates) are divided into bands. Now let L be the unit of exposure. Each band j (j=1..m) has am average common average exposure of v_j = L * j. Each band j is seen as an independent portfolio Denote by k_A the expected loss of obligor A in units of exposure, k_A = v_A/L.Then k, the expected loss in band j (expressed in units of L) can be written k_j = v_j * $lambda_j$ or $lambda_j$ = k_j / v_j.

To analyze the distribution of losses (for each band j) we use a probability generating function G_j for the portfolio defined in terms of an auxiliary variable z (abs(z) <= 1)

$$G_j(z) =$$
$$\sum_{n=0}^{\infty} Pr(\text{Loss} = \text{nL}) * z^n =$$
$$\sum_{n=0}^{\infty} Pr(\#\text{defaults} = \text{n}) * z^{nv_j} =$$
$$\sum_{n=0}^{\infty} \frac{exp(-\lambda_j) * \lambda_j^{\ n}}{n!} * z^{nv_j} =$$
$$exp\left(-\lambda_j + \lambda_j z^{nv_j}\right)$$

The probability generating function for the entire portfolio is
(**Independence** between obligors !!!)

$$G(z) =$$
$$\prod_{j=1}^{m} exp\left(-\lambda_j + \lambda_j z^{v_j}\right) =$$
$$exp\left(-\sum_{j=1}^{m} \lambda_j + \sum_{j=1}^{m} \lambda_j z^{v_j}\right)$$

Now we get as the Loss distribution for the entire portfolio

$$Pr(\text{Loss} = \text{nL}) = \frac{1}{n!} \left. \frac{d^n G(z)}{dz^n} \right|_{z=0} =$$
$$Pr(\text{Loss} = \text{nL}) = \frac{G^{(n)}(0)}{n!}$$

That means that the Loss distribution is “recovered” by taking derivatives from G.

CVaR is now easily derived by first computing the, say, 99 % - percentile and then substracting the expected loss from this number.

The graph below shows a typical Loss distribution.

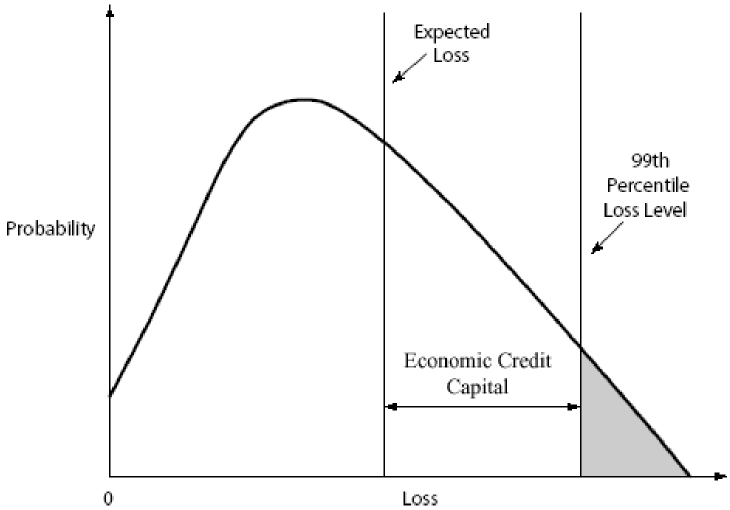

Exhibit CR.1: A typical (Credit) Loss distribution

The principal limitations of this model are that it ignores market and migration risk by focusing only on default events.