Forward Contracts

Consider these scenarios: A farmer wishes to fix the sale price of his crops in advance, an importer arranges to buy foreign currency at a fixed rate in the future, a fund manager who wants to sell stocks for a price known in advance. What can the farmer, the importer or the fund manager do to address their problems?

Each one of them should enter into a forward contract to become independent of the unknown future price of his risky asset (the crops, the foreign currency, the stocks).

So, what exactly is a forward contract? Here comes the definition:

Definition (Forward Contract)
The holder of a long (short) **forward contract** has an agreement to buy (sell) an asset at a certain time in the future for a certain price, which is agreed upon today. The buyer (or seller) in a forward contract:

- acquires a legal obligation to buy (or sell) an asset (known as the underlying asset)
- at some specific future date (the expiration date)
- at a price (the forward price) which is fixed today.

All this new vocabulary related to the forward contracts should become clear by looking at an example:

Example (Forward Contract)
Our farmer wants to lock in a price of $ 50 for a load of crops. He is afraid that the price will drop and he will get less than his $ 50. So he enters into a forward contract (on the underlying asset "load of crops") with the forward price of $ 50 and the expiration date 6 months from now (when harvesting begins).

Actually he is taking a short forward position because he benefits when his underlying asset (the load of crops) goes down in price: If the price drops to $ 40, he still has the right to charge $ 50 for his load. The buyer, e.g. a bakery has the long forward position (The bakery profits when the price of crops goes up) and looses in our scenario.

What happens if the price of crops goes up to $ 60. Here the farmer looses (he has the obligation to sell the crops for $ 50 when the market price is $ 60). On the other hand, the bakery wins for having to pay only $ 50 for the crops which are worth $ 60 now.

You can see that a forward contract is essentially a bet: The farmer bets that the price of the underlyer will not be higher than $ 50, the bakery has the opposite view.

Let us define a new term "payoff of the forward". It's what the forward contract is worth for each party at a moment in time.

In our example we have:

Market price in 6 months	Payoff	
	Long Forward (Bakery)	Short Forward (Farmer)
\$ 35	- \$ 15	\$ 15
\$ 40	- \$ 10	\$ 10
\$ 50	0	0
\$ 60	\$ 10	- \$ 10
\$ 65	\$ 15	- \$ 15

Let us denote the time when the forward contract agreement is made by 0, the expiration date by T, and the forward price by F(0,T). The time t<T the market price of the underlying asset will be denoted by S(t). At expiration the party with a long forward position will benefit if F(0,T) < S(T). (The party can buy the underlying asset for F(0,T) and sell it for the higher market price S(T), making an instant profit of S(T) – F(0,T)). Meanwhile the party holding a short forward position will suffer a loss of S(T) – F(0,T) because they will have to sell below the market price. If F(0,T) > S(T), then the situation is reversed.

If the simultaneous purchase and sale of an asset (in order to profit from a difference in the prices) leads to a difference of 0 then we have a so-called "No-Arbitrage" situation (also known as a "no free lunch" situation). In this case we can compute the prices of different forwards right away.

But at first we introduce some notation. Consider a forward contract established today at time 0. The contract expires at time T (an integer number presenting the number of years). The forward price, which is the price agreed upon today, is denoted as F(0,T). The price of the underlying asset, known as the spot price, is S_0 today, and S_T at the expiration time and in-between at any point in time t S_t. The risk-free interest rate is r_c, compounded annually.

Fair Forward Price of an Asset when there are no Cash Flows on the asset during the Life of the Contract

There are two alternative ways of owning an asset at time T:

1. buy the asset now and hold it until time T, and (Alt.1)
2. take a long position in a forward contract on the asset, that expires at time T (Alt.2)

B/c we have a "No-Arbitrage" situation, the price of 1. , which is S_0 accumulated to time T, or $S_0(1+r)^T$,(where r is the risk-free interest rate compounded annually) must be equal to the price of 2. , which is F(0,T). That leads us to the fair forward price at time T of an asset without any Cash Flows

$$F(0,T) = S_0\,(1 + r)^T \qquad \text{(F.1)}$$

The same thinking leads us to the fair price of a forward contract at any point in time t prior to expiration of the forward contract at time T. We only have to change the accumulation: We don't start the accumulation at time 0 but at time t and apply it to the spot price at t:

$F(t,T) = S_t\ (1 + r)^{T-t}$ (F.2)

Fair Forward Price of an Asset when there are Cash Flows on the asset during the Life of the Contract

Most assets generate cash inflows or outflows (e.g. most stock pay dividends). And that means that the formula (F.1) for the Forward Price doesn't hold anymore: (Alt.2) gets better comparing to (Alt.1) b/c only the holder of the stock gets the corresponding dividend.

In the discrete case we have to substract the present value of the dividends from the spot price S_0. We get

$F(0,T) = \left(S_0 - \sum_{t<T} PV(Div_t) \right)(1 + r_T)^T$. Where PV(.) is the present value operator.

(F.3)

In the continuous case equation (F.3) gets

$$F(0,T) = \left(S_0 - \sum_{i=1}^{n} d_i\ e^{-r_{\tau_i}\ \tau_i} \right) e^{r_T\ T}$$

(F.4)

where we have n points in time $\tau_1, \tau_2, ..., \tau_n \quad (0 < \tau_i < T, \forall i = 1..n)$
and $d_1, d_2, .., d_n$ where at timepoint i the dividend d_i is paid.

Now suppose the underlying asset is a commodity that incurs storage costs that are denoted by γ_i (γ_i is the storage cost for storing commodity i). Then the same thoughts as above leads us to the formula (discrete case)

$$F(0,T) = \left(S_0 - \sum_{i=1}^{n} PV(\gamma_i) \right)(1 + r)^T$$

(F.5)